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Partial Augmented Lagrangian Method and Mathematical Programs with Complementarity Constraints*

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Abstract. In this paper, we apply a partial augmented Lagrangian method to mathematical programs with complementarity constraints (MPCC). Specifically, only the complementarity constraints are incorporated into the objective function of the augmented Lagrangian problem while the other constraints of the original MPCC are retained as constraints in the augmented Lagrangian problem. We show that the limit point of a sequence of points that satisfy second-order necessary conditions of the partial augmented Lagrangian problems is a strongly stationary point (hence a B-stationary point) of the original MPCC if the limit point is feasible to MPCC, the linear independence constraint qualification for MPCC and the upper level strict complementarity condition hold at the limit point. Furthermore, this limit point also satisfies a second-order necessary optimality condition of MPCC. Numerical experiments are done to test the computational performances of several methods for MPCC proposed in the literature.

Key words: B-stationarity, constraint qualification, mathematical programs with complementarity constraints, optimality conditions, partial augmented Lagrangian method

1. Introduction

Consider the following mathematical program with complementarity constraints (MPCC):

$$\begin{array}{ll} \min\limits_{z\in R^n} & f(z) \\ \text{s.t.} & G_i(z)\geqslant 0, \quad H_i(z)\geqslant 0, \quad G_i(z)H_i(z)=0, \quad i=1,\dots,p, \\ & g_j(z)\leqslant 0, \quad j\in J_1, \\ & g_j(z)=0, \quad j\in J_2, \end{array}$$

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where $f, G_i, H_i, g_j : R^n \to R^1$ are all twice continuously differentiable functions and J_1 and J_2 are both finite index sets.

It is known that a mathematical program with equilibrium constraints, which has wide applications in economics and engineering, can be converted into a MPCC problem (see, e.g. [18]). Many authors have studied MPCC. For a comprehensive and in-depth theoretical study of MPCC, we refer the reader to [11, 23, 26, 27] and the references therein. On the other hand, the development of various algorithms for mathematical programs with equilibrium constraints, variational inequality constraints or complementarity constraints can be found in [6,9,10,12–15,17,20,21,24,25] and the references therein.

The augmented Lagrangian method is popular and effective for solving constrained optimization problems (see, e.g. [2]). However, for some constrained optimization problems, it may be more advantageous to employ a partial augmented Lagrangian method, namely, only those constraint functions which are hard to handle will be incorporated into the objective function of the augmented Lagrangian problem while the remaining constraints will be retained explicitly (see, e.g. [2–4,7,8,19]).

Complementarity constraints in MPCC are known to be difficult to treat. In this paper, we shall apply a partial augmented Lagrangian method directly to MPCC. Specifically, only the complementarity constraints are incorporated into the objective function of the augmented Lagrangian problem while the other constraints of the original MPCC are retained as constraints in the augmented Lagrangian problem. We show that the limit point of a sequence of points that satisfy second-order necessary conditions of the partial augmented Lagrangian problems is a strongly stationary point (hence a B-stationary point) of the original MPCC if the limit point is feasible to MPCC, the linear independence constraint qualification for MPCC and the upper level strict complementarity condition hold at the limit point. Furthermore, this limit point also satisfies a second-order necessary optimality condition of MPCC. Numerical experiments will be done to test the computational performances of several methods for MPCC proposed in [10, 13, 14, 24] and this paper.

Denote by Z_0 the feasible set of MPCC, i.e.,

$$Z_0 = \{z \in R^n : G_i(z) \geqslant 0, H_i(z) \geqslant 0, G_i(z)H_i(z) = 0, i = 1, \dots, p,$$
$$g_j(z) \leqslant 0, j \in J_1, g_j(z) = 0, j \in J_2\}.$$

Throughout the paper, we assume that $Z_0 \neq \emptyset$. Let $z \in R^n$. Define

$$I^1(z) = \{i \in \{1, \dots, p\} : G_i(z) = 0, H_i(z) > 0\},$$
$$I^2(z) = \{i \in \{1, \dots, p\} : G_i(z) > 0, H_i(z) = 0\},$$
$$I^3(z) = \{i \in \{1, \dots, p\} : G_i(z) = 0, H_i(z) = 0\},$$

$$I^c(z)=\{1,\dots,p\}\setminus\left[I^1(z)\cup I^2(z)\cup I^3(z)\right],$$
$$J_1(z)=\{j\in J_1:g_j(z)=0\}.$$

Now we give some definitions concerning the first-order and second-order necessary optimality conditions of MPCC.

DEFINITION 1.1 [10]. Let $\bar{z}\in Z_0$. We say that the linear independence constraint qualification (LICQ) for MPCC holds at $\bar{z}$ if

$$\{\nabla G_i(\bar{z}):i\in I^1(\bar{z})\cup I^3(\bar{z})\}\cup\{\nabla H_i(\bar{z}):i\in I^2(\bar{z})\cup I^3(\bar{z})\}$$
$$\cup\{\nabla g_j(\bar{z}):j\in J_1(\bar{z})\}\cup\{\nabla g_j(\bar{z}):j\in J_2\}$$

are linearly independent.

DEFINITION 1.2 [26]. Let $\bar{z}\in Z_0$. We say that $\bar{z}$ is a strongly stationary point of MPCC if the following conditions hold at $\bar{z}$:

$$\nabla f(\bar{z})+\sum_{i\in I^1(\bar{z})}v_i\nabla G_i(\bar{z})+\sum_{i\in I^2(\bar{z})}w_i\nabla H_i(\bar{z})+\sum_{i\in I^3(\bar{z})}(v_i\nabla G_i(\bar{z})+w_i\nabla H_i(\bar{z}))$$
$$+\sum_{j\in J_1(\bar{z})}\mu_j\nabla g_j(\bar{z})+\sum_{j\in J_2}\nu_j\nabla g_j(\bar{z})=0,$$
$$v_i,w_i\leqslant 0,\quad i\in I^3(\bar{z}),\tag{1}$$
$$\mu_j\geqslant 0,\quad j\in J_1(\bar{z}).\tag{2}$$

DEFINITION 1.3 [1]. Let $\bar{z}\in Z_0$. The contingent tangent cone of Z_0 at $\bar{z}$ is defined as

$$T_{Z_0}(\bar{z})=\left\{d\in R^n:\exists t_k\downarrow 0\text{ and }z^k\in Z_0\text{ such that }\lim_{k\to+\infty}\frac{z^k-\bar{z}}{t^k}=d\right\}.$$

DEFINITION 1.4 [18]. Let $\bar{z}\in Z_0$. $\bar{z}$ is called a B-stationary point of MPCC if

$$\nabla f(\bar{z})^T d\geqslant 0,\quad \forall d\in T_{Z_0}(\bar{z}).$$

Here we use the definition of a B-stationary point given in [18]. Another definition of a B-stationary point was given in [26]. Obviously, a B-stationary point in the sense of [26] is a B-stationary point in [18]. Moreover, it is clear from [26] that a strongly stationary point of MPCC is a B-stationary point in the sense of [26], hence a B-stationary point in [18].

However, all these three concepts of stationarity are equivalent if LICQ for MPCC holds at $\bar{z}\in Z_0$.

It is clear from [26] that if $\bar{z}$ is a local minimum of MPCC and LICQ for MPCC holds at $\bar{z}$, then $\bar{z}$ is a strongly stationary point of MPCC, and hence a B-stationary point.

DEFINITION 1.5 [14]. Let $\bar{z}\in Z_0$. We say that a second-order condition of MPCC is satisfied at $\bar{z}$ if $\bar{z}$ is strongly stationary point, i.e. (1) and (2) holds, and for every vector $d\in R^n$ such that

$$\begin{aligned}
&\nabla G_i(\bar{z})^T d=0, \quad i\in I^1(\bar{z}),\\
&\nabla H_i(\bar{z})^T d=0, \quad i\in I^2(\bar{z}),\\
&\nabla G_i(\bar{z})^T d=0, \quad i\in I^3(\bar{z}),\\
&\nabla H_i(\bar{z})^T d=0, \quad i\in I^3(\bar{z}),\\
&\nabla g_j(\bar{z})^T d=0, \quad j\in J_1(\bar{z}),\\
&\nabla g_j(\bar{z})^T d=0, \quad j\in J_2,
\end{aligned} \tag{3}$$

there holds

$$\begin{aligned}
d^T\Bigg[&\nabla^2 f(\bar{z})+\sum_{i\in I^1(\bar{z})} v_i\,\nabla^2 G_i(\bar{z})+\sum_{i\in I^2(\bar{z})} w_i\,\nabla^2 H_i(\bar{z})\\
&+\sum_{i\in I^3(\bar{z})}(v_i\,\nabla^2 G_i(\bar{z})+w_i\,\nabla^2 H_i(\bar{z}))\\
&+\sum_{j\in J_1(\bar{z})}\mu_j\,\nabla^2 g_j(\bar{z})+\sum_{j\in J_2}^{q}\mu_j\,\nabla^2 g_j(\bar{z})\Bigg]d\geqslant 0.
\end{aligned} \tag{4}$$

DEFINITION 1.6 [24]. Let $\bar{z}\in Z_0$. Assume that (1) holds. We say that upper level strict complementarity condition (ULSC) holds at $\bar{z}$ if $v_i w_i\neq 0$, $\forall i\in I_2^3(\bar{z})$.

2. A Partial Augmented Lagrangian Method for MPCC

Consider the following partial augmented Lagrangian problem $(P(y,r))$:

$$\begin{aligned}
\min\quad & L(z,y,r)\\
\text{s.t.}\quad & G_i(z)\geqslant 0,\quad H_i(z)\geqslant 0,\quad i=1,\dots,p,\\
& g_j(z)\leqslant 0,\quad j\in J_1,\\
& g_j(z)=0,\quad j\in J_2,
\end{aligned}$$

where

$$L(z,y,r)=f(z)+\sum_{i=1}^{p}y_iG_i(z)H_i(z)+r/2\sum_{i=1}^{p}[G_i(z)H_i(z)]^2,$$
$$z\in R^n,\quad y\in R^p,\ r>0.$$

It is routine to derive the following necessary conditions for a local minimum of $(P(y,r))$.

PROPOSITION 2.1 Let $\bar{z}$ be a local optimal solution of $(P(y,r))$. Suppose that the following condition (C) holds:

$$\left\{\nabla G_i(\bar{z}):i\in I^1(\bar{z})\cup I^3(\bar{z})\right\}\cup\left\{\nabla H_i(\bar{z}):i\in I^2(\bar{z})\cup I^3(\bar{z})\right\}\cup \left\{\nabla g_j(\bar{z}):j\in J_1(\bar{z})\cup J_2\right\}$$

are linearly independent.

Then, the following first-order necessary condition holds:
there exist $v_i\leqslant 0, i\in I^1(\bar{z})\cup I^3(\bar{z}),\ w_i\leqslant 0, i\in I^2(\bar{z})\cup I^3(\bar{z}),\ \mu_j\geqslant 0, j\in J_1(\bar{z})$ and $\mu_j,\ j\in J_2$ such that

$$\begin{aligned}\nabla f(\bar{z})&+\sum_{i\in I^c(\bar{z})}[H_i(\bar{z})(y_i+rG_i(\bar{z})H_i(\bar{z}))]\nabla G_i(\bar{z})\\&+\sum_{i\in I^c(\bar{z})}[G_i(\bar{z})(y_i+rG_i(\bar{z})H_i(\bar{z}))]\nabla H_i(\bar{z})\\&+\sum_{i\in I^1(\bar{z})\cup I^3(\bar{z})}\left[y_iH_i(\bar{z})+rG_i(\bar{z})H_i^2(\bar{z})+v_i\right]\nabla G_i(\bar{z})\\&+\sum_{i\in I^2(\bar{z})\cup I^3(\bar{z})}\left[y_iG_i(\bar{z})+rH_i(\bar{z})G_i^2(\bar{z})+w_i\right]\nabla H_i(\bar{z})\\&+\sum_{j\in J_1(\bar{z})\cup J_2}\mu_j\nabla g_j(\bar{z})=0\end{aligned}\tag{5}$$

and the second-order necessary condition holds:
the first-order condition holds and for any $d\in R^n$ satisfying

$$\begin{aligned}&\nabla G_i(\bar{z})^Td=0,\quad i\in I^1(\bar{z}),\\&\nabla H_i(\bar{z})^Td=0,\quad i\in I^2(\bar{z}),\\&\nabla G_i(\bar{z})^Td=0,\quad i\in I^3(\bar{z}),\\&\nabla H_i(\bar{z})^Td=0,\quad i\in I^3(\bar{z}),\\&\nabla g_j(\bar{z})^Td=0,\quad j\in J_1(\bar{z})\cup J_2,\end{aligned}\tag{6}$$

there holds

$$
\begin{aligned}
& d^T \nabla^2 f(\bar{z})d + \sum_{i\in I^c(\bar{z})} [H_i(\bar{z})(y_i + rG_i(\bar{z})H_i(\bar{z}))]\, d^T \nabla^2 G_i(\bar{z})d \\
&\quad + \sum_{i\in I^c(\bar{z})} [G_i(\bar{z})(y_i + rG_i(\bar{z})H_i(\bar{z}))]\, d^T \nabla^2 H_i(\bar{z})d \\
&\quad + \sum_{i\in I^1(\bar{z})\cup I^3(\bar{z})} \left[y_i H_i(\bar{z}) + rG_i(\bar{z})H_i^2(\bar{z}) + v_i\right] d^T \nabla^2 G_i(\bar{z})d \\
&\quad + \sum_{i\in I^2(\bar{z})\cup I^3(\bar{z})} \left[y_i G_i(\bar{z}) + rH_i(\bar{z})G_i^2(\bar{z}) + w_i\right] d^T \nabla^2 H_i(\bar{z})d \\
&\quad + \sum_{j\in J_1(\bar{z})\cup J_2} \mu_j d^T \nabla^2 g_j(\bar{z})d \\
&\quad +2\sum_{i=1}^{p} (y_i + rG_i(\bar{z})H_i(\bar{z}))(\nabla G_i(\bar{z})d)(\nabla H_i(\bar{z})d) \\
&\quad +r\sum_{i=1}^{p} [H_i(\bar{z})(\nabla G_i(\bar{z})d) + G_i(\bar{z})(\nabla H_i(\bar{z})d)]^2 \geqslant 0. \qquad (7)
\end{aligned}
$$

3. Convergence Results

THEOREM 3.1 Suppose that $\{y^k\} \subseteq R^p$ is a bounded sequence and $0 < r_k \to +\infty$. Let each $\bar{z}_k$ be feasible to $(P(y^k, r_k))$ and satisfy the first-order necessary optimality condition of $(P(y^k, r_k))$. Assume that there exists a real number M such that

$$
L(\bar{z}_k, y^k, r_k) \leqslant M, \quad \forall k. \qquad (8)
$$

Suppose that $\bar{z}$ is a limit point of $\{\bar{z}_k\}$. Then $\bar{z}$ is feasible to the original MPCC. Futhermore, if the LICQ for MPCC holds at $\bar{z}$, then there exist $\bar{v}_i, i \in I^1(\bar{z}) \cup I^3(\bar{z})$, $\bar{w}_i, i \in I^2(\bar{z}) \cup I^3(\bar{z})$, $\bar{\mu}_j \geqslant 0, j \in J_1(\bar{z})$, $\bar{\mu}_j, j \in J_2$ such that

$$
\begin{aligned}
& \nabla f(\bar{z}) + \sum_{i\in I^1(\bar{z})} \bar{v}_i \nabla G_i(\bar{z}) + \sum_{i\in I^2(\bar{z})} \bar{w}_i \nabla H_i(\bar{z}) \\
&\quad + \sum_{i\in I^3(\bar{z})} (\bar{v}_i \nabla G_i(\bar{z}) + \bar{w}_i \nabla H_i(\bar{z})) \\
&\quad + \sum_{j\in J_1(\bar{z})\cup J_2} \bar{\mu}_j \nabla g_j(\bar{z}) = 0. \qquad (9)
\end{aligned}
$$

Proof. Assume without loss of generality that $\bar{z}_k \to \bar{z}$ as $k \to +\infty$. Since each $\bar{z}_k$ is feasible to $(P(y^k, r_k))$, we have

$$\begin{aligned} &G_i(\bar{z}_k) \geqslant 0, \quad H_i(\bar{z}_k) \geqslant 0, \quad i = 1, \dots, p, \\ &g_j(\bar{z}_k) \leqslant 0, \quad j \in J_1, \\ &g_j(\bar{z}_k) = 0, \quad j \in J_2. \end{aligned}$$

Passing to the limit as $k \to +\infty$, we get

$$\begin{aligned} &G_i(\bar{z}) \geqslant 0, \quad H_i(\bar{z}) \geqslant 0, \quad i = 1, \dots, p, \\ &g_j(\bar{z}) \leqslant 0, \quad j \in J_1, \\ &g_j(\bar{z}) = 0, \quad j \in J_2. \end{aligned} \tag{10}$$

Furthermore, from the boundedness of $\{y^k\}$, (8) and the fact that $\bar{z}_k \to \bar{z}$, we see that there exists $M' > 0$ such that

$$r_k/2 \sum_{i=1}^{p} G_i^2(\bar{z}_k) H_i^2(\bar{z}_k) \leqslant M',$$

i.e.,

$$\sum_{i=1}^{p} G_i^2(\bar{z}_k) H_i^2(\bar{z}_k) \leqslant 2M'/r_k.$$

Passing to the limit as $k \to +\infty$, we have

$$\sum_{i=1}^{p} G_i^2(\bar{z}) H_i^2(\bar{z}) = 0.$$

Namely,

$$G_i(\bar{z}) H_i(\bar{z}) = 0, \quad i = 1, \dots, p.$$

This combined with (11) yields that $\bar{z} \in Z_0$. That is, $\bar{z}$ is feasible to MPCC. As each $\bar{z}_k$ satisfies the first-order necessary optimality condition of $(P(y^k, r_k))$, we have $v_i^k \leqslant 0, i \in I^1(\bar{z}_k) \cup I^3(\bar{z}_k)$, $w_i^k \leqslant 0, i \in I^2(\bar{z}_k) \cup I^3(\bar{z}_k)$, $\mu_j^k \geqslant 0$, $j \in J_1(\bar{z}_k)$ and μ_j^k, $j \in J_2$ such that

$$\begin{aligned} \nabla f(\bar{z}_k) &+ \sum_{i \in I^c(\bar{z}_k)} \left[H_i(\bar{z}_k)(y_i^k + r_k G_i(\bar{z}_k) H_i(\bar{z}_k)) \right] \nabla G_i(\bar{z}_k) \\ &+ \sum_{i \in I^c(\bar{z}_k)} \left[G_i(\bar{z}_k)(y_i^k + r_k G_i(\bar{z}_k) H_i(\bar{z}_k)) \right] \nabla H_i(\bar{z}_k) \end{aligned}$$

$$+\sum_{i\in I^1(\bar z_k)\bigcup I^3(\bar z_k)}\left[y_i^k H_i(\bar z_k)+r_k G_i(\bar z_k)H_i^2(\bar z_k)+v_i^k\right]\nabla G_i(\bar z_k)$$
$$+\sum_{i\in I^2(\bar z_k)\bigcup I^3(\bar z_k)}\left[y_i^k G_i(\bar z_k)+r_k H_i(\bar z_k)G_i^2(\bar z_k)+w_i^k\right]\nabla H_i(\bar z_k)$$
$$+\sum_{j\in J_1(\bar z_k)\bigcup J_2}\mu_j^k\nabla g_j(\bar z_k)=0. \tag{11}$$

Let

$$\begin{aligned}
&\bar v_i^k=H_i(\bar z_k)(y_i^k+r_kG_i(\bar z_k)H_i(\bar z_k)), \quad i\in I^c(\bar z_k),\\
&\bar w_i^k=G_i(\bar z_k)(y_i^k+r_kG_i(\bar z_k)H_i(\bar z_k)), \quad i\in I^c(\bar z_k),\\
&\bar v_i^k=H_i(\bar z_k)(y_i^k+r_kG_i(\bar z_k)H_i(\bar z_k))+v_i^k, \quad i\in I^1(\bar z_k)\cup I^3(\bar z_k),\\
&\bar w_i^k=0, \quad i\in I^1(\bar z_k),\\
&\bar v_i^k=0, \quad i\in I^2(\bar z_k),\\
&\bar w_i^k=G_i(\bar z_k)(y_i^k+r_kG_i(\bar z_k)H_i(\bar z_k))+w_i^k, \quad i\in I^2(\bar z_k)\cup I^3(\bar z_k),\\
&\bar\mu_j^k=\mu_j^k\geqslant 0, \quad j\in J_1(\bar z_k),\\
&\bar\mu_j^k=0, \quad j\in J_1(\bar z)\setminus J_1(\bar z_k),\\
&\bar\mu_j^k=\mu_j^k, \quad j\in J_2.
\end{aligned} \tag{12}$$

Obviously, we can assume without loss of generality that

$$J_1(\bar z_k)\subseteq J_1(\bar z), \quad \forall k. \tag{13}$$

Substituting (13) into (11) while observing (13), we have

$$\nabla f(\bar z_k)+\sum_{i=1}^p(\bar v_i^k\nabla G_i(\bar z_k)+\bar w_i^k\nabla H_i(\bar z_k))+\sum_{j\in J_1(\bar z)}\bar\mu_j^k\nabla g_j(\bar z_k)$$
$$+\sum_{j\in J_2}\bar\mu_j^k\nabla g_j(\bar z_k)=0. \tag{14}$$

Let

$$\tau_k=\sum_{i=1}^p(|\bar v_i^k|+|\bar w_i^k|)+\sum_{j\in J_1(\bar z)}\bar\mu_j^k+\sum_{j\in J_2}|\bar\mu_j^k|.$$

We show by contradiction that $\{\tau_k\}$ is bounded. Otherwise, we assume, without loss of generality, that $\tau_k\to+\infty$ and

$$\begin{aligned}
&\bar v_i^k/\tau_k\to\bar v_i', \quad i=1,\ldots,p,\\
&\bar w_i^k/\tau_k\to\bar w_i', \quad i=1,\ldots,p,
\end{aligned}$$

$$\bar{\mu}_j^k/\tau_k \to \bar{\mu}'_j \geqslant 0, \quad j \in J_1(\bar{z}),$$
$$\bar{\mu}_j^k/\tau_k \to \bar{\mu}'_j, \quad j \in J_2. \tag{15}$$

Dividing (14) by τ_k and passing to the limit as $k \to +\infty$, we have

$$\sum_{i=1}^{p}(\bar{v}'_i \nabla G_i(\bar{z}) + \bar{w}'_i \nabla H_i(\bar{z})) + \sum_{j \in J_1(\bar{z}) \cup J_2} \bar{\mu}'_j \nabla g_j(\bar{z}) = 0. \tag{16}$$

Moreover, we have

$$\sum_{i=1}^{p}(|\bar{v}'_i| + |\bar{w}'_i|) + \sum_{j \in J_1(\bar{z})} \bar{\mu}'_j + \sum_{j \in J_2} |\bar{\mu}'_j| = 1. \tag{17}$$

We show that

$$\bar{w}'_i = 0, \quad i \in I^1(\bar{z}) \tag{18}$$

and

$$v'_i = 0, \quad i \in I^2(\bar{z}). \tag{19}$$

We prove only (18) and (19) can be analogously proved. Suppose that $i \in I^1(\bar{z})$. Then from (13) and the fact that $\bar{z}_k \to \bar{z}$, we deduce that $i \in I^1(\bar{z}_k)$ or $i \in I^c(\bar{z}_k)$ when k is sufficiently large. Consider the following two cases.

(i) There exist infinitely many k's such that $i \in I^1(\bar{z}_k)$.
(ii) $i \in I^c(\bar{z}_k), \quad k \geqslant k_0$ for some $k_0 > 0$.

If case (i) is true, we assume without loss of generality that $i \in I^1(\bar{z}_k), \quad k \geqslant k_1$ for some $k_1 > 0$. As a result,

$$\bar{w}'_i = \lim_{k \to +\infty} \frac{\bar{w}_i^k}{\tau_k} = \lim_{k \to +\infty} 0 = 0.$$

Thus (18) holds. If case (ii) is true, then

$$|\bar{w}'_i| = \lim_{k \to +\infty} \left|\frac{\bar{w}_i^k}{\tau_k}\right| \leqslant \lim_{k \to +\infty} \left|\frac{\bar{w}_i^k}{\bar{v}_i^k}\right| = \lim_{k \to +\infty} \frac{G_i(\bar{z}_k)}{H_i(\bar{z}_k)} = 0,$$

i.e. (18) holds.

Substituting (18) and (19) into (16) and (17), we obtain

$$\sum_{i \in I^1(\bar{z})} \bar{v}'_i \nabla G_i(\bar{z}) + \sum_{i \in I^2(\bar{z})} \bar{w}'_i \nabla H_i(\bar{z}) + \sum_{i \in I^3(\bar{z})} (\bar{v}'_i \nabla G_i(\bar{z}) + \bar{w}'_i \nabla H_i(\bar{z}))$$
$$+ \sum_{j \in J_1(\bar{z}) \bigcup J_2} \bar{\mu}'_j \nabla g_j(\bar{z}) = 0 \tag{20}$$

and

$$\sum_{i\in I^1(\bar{z})}|\bar{v}_i'|+\sum_{i\in I^2(\bar{z})}|\bar{w}_i'|+\sum_{i\in I^3(\bar{z})}(|\bar{v}_i'|+|\bar{w}_i'|)+\sum_{j\in J_1(\bar{z})}\bar{\mu}_j'+\sum_{j\in J_2}|\bar{\mu}_j'|=1. \tag{21}$$

The combination of (20) and (21) contradicts the LICQ for MPCC at $\bar{z}$. Hence, $\{\tau_k\}$ is bounded. Consequently, we can assume without loss of generality that

$$\begin{aligned}
&\bar{v}_i^k\to\bar{v}_i, \quad i=1,\dots,p,\\
&\bar{w}_i^k\to\bar{w}_i, \quad i=1,\dots,p,\\
&\bar{\mu}_j^k\to\bar{\mu}_j\geqslant 0, \quad j\in J_1(\bar{z}),\\
&\bar{\mu}_j^k\to\bar{\mu}_j, \quad j\in J_2.
\end{aligned} \tag{22}$$

Taking the limit in (14) as $k\to+\infty$, we get

$$\nabla f(\bar{z})+\sum_{i=1}^{p}(\bar{v}_i\nabla G_i(\bar{z})+\bar{w}_i\nabla H_i(\bar{z}))+\sum_{j\in J_1(\bar{z})\cup J_2}\bar{\mu}_j\nabla g_j(\bar{z})=0. \tag{23}$$

Now we show that

$$\bar{w}_i=0, \quad i\in I^1(\bar{z}) \tag{24}$$

and

$$\bar{v}_i=0, \quad i\in I^2(\bar{z}). \tag{25}$$

We need only to prove (24) since (25) can be similarly proved. As $\lim_{k\to+\infty}\bar{v}_i^k=\bar{v}_i$ and $\lim_{k\to+\infty}H_i(\bar{z}_k)=H_i(\bar{z})$, it follows from (13) that

$$\lim_{k\to+\infty}\left(y_i^k+r_kG_i(\bar{z}_k)H_i(\bar{z}_k)\right)=\lim_{k\to+\infty}\frac{\bar{v}_i^k}{H_i(\bar{z}_k)}=\frac{\bar{v}_i}{H_i(\bar{z})}.$$

Hence,

$$\bar{w}_i=\lim_{k\to+\infty}G_i(\bar{z}_k)\left(y_i^k+r_kG_i(\bar{z}_k)H_i(\bar{z}_k)\right)=G_i(\bar{z})\cdot\frac{\bar{v}_i}{H_i(\bar{z})}=0.$$

Substituting (24) and (25) into (23), we obtain (9) and the proof is complete. □

We need the next lemma to prove further convergence results.

LEMMA 3.1 Let $\{c_i^k\}_{k=1}^{\infty} \subseteq R^n, i = 1, \ldots, s$ be sequences such that

$$\lim_{k\to+\infty} c_i^k = c_i, \quad i = 1, \ldots, s.$$

Suppose that $\{c_i : i = 1, \ldots, s\}$ are linearly independent. Then $\forall \bar{d} \in \{d \in R^n : c_i^T d = 0, i = 1, \ldots, s\}$, there exists $\bar{k} > 0$ such that, when $k \geqslant \bar{k}$, there exists $d^k \in R^n$ satisfying $c_i^k d^k = 0, \quad i = 1, \ldots, s$ and $d^k \to \bar{d}$.

Proof. It follows directly from ([5], Corollary II.3.4) (see also ([31], Lemma 5.1). □

THEOREM 3.2 Let the assumptions of Theorem 3.1 hold. Further assume that the ULSC holds at $\bar{z}$ and the second-order necessary condition of $(P(y^k, r_k))$ holds at $\bar{z}_k$ (see Proposition 2.1). Then $\bar{z}$ is a strongly stationary point of MPCC. Moreover, the second-order condition (in Definition 1.5) of MPCC also holds at $\bar{z}$.

Proof. First we prove that $\bar{z}$ is a strongly stationary point of MPCC. Suppose to the contrary that there exists $i^* \in I^3(\bar{z})$ such that $\bar{v}_{i^*} > 0$. Then, by the ULSC condition, we deduce that $\bar{w}_{i^*} \neq 0$. From (11), (13) and the fact that $v_i^k \leqslant 0, i \in I^1(\bar{z}_k) \cup I^3(\bar{z}_k)$, $w_i^k \leqslant 0, i \in I^2(\bar{z}_k) \cup I^3(\bar{z}_k)$ when k is sufficiently large, we deduce that $i^* \notin I^1(\bar{z}_k) \cup I^2(\bar{z}_k) \cup I^3(\bar{z}_k)$. Consequently, we must have $i^* \in I^c(\bar{z}_k)$. This combined with (13) and (22) yields

$$\begin{aligned} \bar{v}_{i^*} &= \lim\nolimits_{k\to+\infty} H_{i^*}(\bar{z}_k) \left(y_{i^*}^k + r_k G_{i^*}(\bar{z}_k) H_{i^*}(\bar{z}_k)\right) > 0, \\ \bar{w}_{i^*} &= \lim\nolimits_{k\to+\infty} G_{i^*}(\bar{z}_k) \left(y_{i^*}^k + r_k G_{i^*}(\bar{z}_k) H_{i^*}(\bar{z}_k)\right) > 0. \end{aligned} \tag{26}$$

In particular, we should have

$$\lim_{k\to+\infty} \left(y_{i^*}^k + r_k G_{i^*}(\bar{z}_k) H_{i^*}(\bar{z}_k)\right) = +\infty. \tag{27}$$

By the second-order necessary condition of $(P(y^k, r_k))$ at $\bar{z}_k$ and (13), we see that for any d satisfying

$$\begin{aligned} &\nabla G_i(\bar{z}_k)^T d = 0, \quad i \in I^1(\bar{z}_k), \\ &\nabla H_i(\bar{z}_k)^T d = 0, \quad i \in I^2(\bar{z}_k), \\ &\nabla G_i(\bar{z}_k)^T d = 0, \quad i \in I^3(\bar{z}_k), \\ &\nabla H_i(\bar{z}_k)^T d = 0, \quad i \in I^3(\bar{z}_k), \\ &\nabla g_j(\bar{z}_k)^T d = 0, \quad j \in J_1(\bar{z}_k) \cup J_2, \end{aligned} \tag{28}$$

there holds

$$\begin{aligned}
&d^T \nabla^2 f(\bar{z}_k)d + \sum_{i=1}^{p} \left(\bar{v}_i^k d^T \nabla^2 G_i(\bar{z}_k)d + \bar{w}_i^k d^T \nabla^2 H_i(\bar{z}_k)d\right) \\
&\quad + \sum_{j\in J_1(\bar{z})\cup J_2} \mu_j d^T \nabla^2 g_j(\bar{z})d \\
&\quad +2 \sum_{i\in I^c(\bar{z}_k)\setminus\{i^*\}} (y_i^k + r_k G_i(\bar{z}_k)H_i(\bar{z}_k))(\nabla G_i(\bar{z}_k)d)(\nabla H_i(\bar{z}_k)d) \\
&\quad +r_k \sum_{i\in I^c(\bar{z}_k)\setminus\{i^*\}} [H_i(\bar{z}_k)(\nabla G_i(\bar{z}_k)d) + G_i(\bar{z}_k)(\nabla H_i(\bar{z}_k)d)]^2 \\
&\quad +2(y_{i^*}^k + r_k G_{i^*}(\bar{z}_k)H_{i^*}(\bar{z}_k))(\nabla G_{i^*}(\bar{z}_k)d)(\nabla H_{i^*}(\bar{z}_k)d) \\
&\quad +r_k [H_{i^*}(\bar{z}_k)(\nabla G_{i^*}(\bar{z}_k)d) + G_{i^*}(\bar{z}_k)(\nabla H_{i^*}(\bar{z}_k)d)]^2 \geqslant 0.
\end{aligned} \tag{29}$$

By the LICQ of MPCC at $\bar{z}$ and the fact that $\bar{z}_k \to \bar{z}$, we can choose $\{d_k\} \subseteq R^n$ such that $\{d_k\}$ is bounded and

$$\begin{aligned}
&\nabla G_{i^*}(\bar{z}_k)^T d_k = \frac{G_{i^*}(\bar{z}_k)}{G_{i^*}(\bar{z}_k) + H_{i^*}(\bar{z}_k)}, \\
&\nabla H_{i^*}(\bar{z}_k)^T d_k = -\frac{H_{i^*}(\bar{z}_k)}{G_{i^*}(\bar{z}_k) + H_{i^*}(\bar{z}_k)}, \\
&\nabla G_i(\bar{z}_k)^T d_k = \nabla H_i(\bar{z}_k) d_k = 0, \quad i \in \left[I^3(\bar{z}) \cap I^c(\bar{z}_k)\right] \setminus \{i^*\}, \\
&\nabla G_i(\bar{z}_k)^T d_k = 0, \quad i \in I^c(\bar{z}_k) \cap I^1(\bar{z}_k), \\
&\nabla H_i(\bar{z}_k)^T d_k = 0, \quad i \in I^c(\bar{z}_k) \cap I^2(\bar{z}_k), \\
&\nabla G_i(\bar{z}_k)^T d_k = 0, \quad i \in I^1(\bar{z}_k) \cap I^3(\bar{z}_k), \\
&\nabla H_i(\bar{z}_k)^T d_k = 0, \quad i \in I^2(\bar{z}_k) \cap I^3(\bar{z}_k), \\
&\nabla g_j(\bar{z}_k)^T d_k = 0, \quad j \in J_1(\bar{z}_k) \cup J_2.
\end{aligned} \tag{30}$$

Note that each d_k satisfying (30) meets (28). Substituting (30) into (29) (with d replaced by d_k), we obtain

$$\begin{aligned}
&d_k^T \nabla^2 f(\bar{z}_k)d_k + \sum_{i=1}^{p} \left(\bar{v}_i^k d_k^T \nabla^2 G_i(\bar{z}_k)d_k + \bar{w}_i^k d_k^T \nabla^2 H_i(\bar{z}_k)d_k\right) \\
&\quad + \sum_{j\in J_1(\bar{z})\cup J_2} \mu_j d_k^T \nabla^2 g_j(\bar{z})d_k \\
&\quad +r_k \sum_{i\in I^c(\bar{z}_k)\cap I^1(\bar{z}_k)} G_i^2(\bar{z}_k)\left(\nabla H_i(\bar{z}_k)^T d_k\right)^2
\end{aligned}$$

$$+r_k \sum_{i\in I^c(\bar z_k)\cap I^2(\bar z_k)} H_i^2(\bar z_k)\left(\nabla G_i(\bar z_k)^T d_k\right)^2$$
$$-2(y_{i^*}^k + r_k G_{i^*}(\bar z_k)H_{i^*}(\bar z_k))\frac{G_{i^*}(\bar z_k)H_{i^*}(\bar z_k)}{(G_{i^*}(\bar z_k)+H_{i^*}(\bar z_k))^2} \geqslant 0. \tag{31}$$

Due to (13), (22) and the boundedness of $\{y^k\}$, it is easily checked that the sequences $\{r_k G_i^2(\bar z_k)(\nabla H_i(\bar z_k)d_k)^2\}$ and $\{r_k H_i^2(\bar z_k)(\nabla G_i(\bar z_k)d_k)^2\}$ are bounded. As a result, all the terms except the last one on the left-hand side of (31) are bounded as $k\to+\infty$. Moreover, applying (13) and (22), we have

$$\lim_{k\to+\infty}\frac{G_{i^*}(\bar z_k)H_{i^*}(\bar z_k)}{(G_{i^*}(\bar z_k)+H_{i^*}(\bar z_k))^2}$$
$$=\lim_{k\to+\infty}\frac{\left[G_{i^*}(\bar z_k)(y_{i^*}^k+r_kG_{i^*}(\bar z_k)H_{i^*}(\bar z_k))\right]\left[H_{i^*}(\bar z_k)(y_{i^*}^k+r_kG_{i^*}(\bar z_k)H_{i^*}(\bar z_k))\right]}{\left[G_{i^*}(\bar z_k)(y_{i^*}^k+r_kG_{i^*}(\bar z_k)H_{i^*}(\bar z_k))+H_{i^*}(\bar z_k)(y_{i^*}^k+r_kG_{i^*}(\bar z_k)H_{i^*}(\bar z_k))\right]^2}$$
$$=\frac{\bar v_{i^*}\bar w_{i^*}}{(\bar v_{i^*}+\bar w_{i^*})^2}>0.$$

This together with (27) implies that the last term on the left-hand side of (31) tends to $-\infty$ as $k\to+\infty$. It follows that the inequality (31) is impossible as $k\to+\infty$. So we must have $\bar v_i\leqslant 0, \bar w_i\leqslant 0, \forall i\in I^3(\bar z)$.

Finally, we prove that $\bar z$ satisfies the second-order necessary optimality condition of MPCC.

Suppose that $d\in R^n$ satisfies (3). Note that

$$\begin{aligned}
&\nabla G_i(\bar z_k)\to\nabla G_i(\bar z), \quad i\in\left[I^c(\bar z_k)\cup I^1(\bar z_k)\right]\cap I^1(\bar z),\\
&\nabla H_i(\bar z_k)\to\nabla H_i(\bar z), \quad i\in\left[I^c(\bar z_k)\cup I^2(\bar z_k)\right]\cap I^2(\bar z),\\
&\nabla G_i(\bar z_k)\to\nabla G_i(\bar z), \quad i\in\left[I^1(\bar z_k)\cup I^2(\bar z_k)\cup I^3(\bar z_k)\right]\cap I^3(\bar z),\\
&\nabla H_i(\bar z_k)\to\nabla H_i(\bar z), \quad i\in\left[I^1(\bar z_k)\cup I^2(\bar z_k)\cup I^3(\bar z_k)\right]\cap I^3(\bar z),\\
&\nabla G_i(\bar z_k)\to\nabla G_i(\bar z), \quad \nabla H_i(\bar z_k)\to\nabla H_i(\bar z), \quad i\in I^c(\bar z_k)\cap I^3(\bar z),\\
&\nabla g_j(\bar z_k)\to\nabla g_j(\bar z), \quad j\in J_1(\bar z)\cup J_2.
\end{aligned}$$

Note also that when k is sufficiently large, we have

$$\begin{aligned}
&I^1(\bar z)\subseteq I^1(\bar z_k)\cap I^c(\bar z_k),\\
&I^2(\bar z)\subseteq I^2(\bar z_k)\cap I^c(\bar z_k).
\end{aligned}$$

By the LICQ of MPCC at $\bar z$ and Lemma 3.1, we can find $\{d_k\}\subseteq R^n$ such that

(i) $d_k\to d$;

(ii)

$$\begin{aligned}
&\nabla G_i(\bar{z}_k)^T d_k = 0, \quad i \in \left[I^c(\bar{z}_k) \cup I^1(\bar{z}_k)\right] \cap I^1(\bar{z}),\\
&\nabla H_i(\bar{z}_k)^T d_k = 0, \quad i \in \left[I^c(\bar{z}_k) \cup I^2(\bar{z}_k)\right] \cap I^2(\bar{z}),\\
&\nabla G_i(\bar{z}_k)^T d_k = \nabla H_i^T(\bar{z}_k) d_k = 0,\\
&\quad i \in \left[I^c(\bar{z}_k) \cup \left(I^1(\bar{z}_k) \cup I^2(\bar{z}_k) \cup I^3(\bar{z}_k)\right)\right] \cap I^3(\bar{z}),\\
&\nabla g_j(\bar{z}_k)^T d_k = 0, \quad j \in J_1(\bar{z}) \cup J_2.
\end{aligned} \tag{32}$$

It is clear that any d_k satisfying (32) also satisfies (28) (with d replaced by d_k). Substituting (32) into (29) (with d replaced by d_k), we get

$$\begin{aligned}
d_k^T \nabla^2 f(\bar{z}_k) d_k + \sum_{i=1}^{p} &\left(\bar{v}_i^k d_k^T \nabla^2 G_i(\bar{z}_k) d_k + \bar{w}_i^k d_k^T \nabla^2 H_i(\bar{z}_k) d_k\right)\\
&+ \sum_{j \in J_1(\bar{z}) \bigcup J_2} \mu_j d_k^T \nabla^2 g_j(\bar{z}) d_k\\
&+ r_k \sum_{i \in I^c(\bar{z}_k) \bigcap I^1(\bar{z})} G_i^2(\bar{z}_k) \left(\nabla H_i(\bar{z}_k)^T d_k\right)^2\\
&+ r_k \sum_{i \in I^c(\bar{z}_k) \bigcap I^2(\bar{z})} H_i^2(\bar{z}_k) \left(\nabla G_i(\bar{z}_k)^T d_k\right)^2 \geqslant 0.
\end{aligned} \tag{33}$$

Moreover, from (13), (22) and the boundedness of $\{y^k\}$, we deduce that

$$\begin{aligned}
&\lim_{k \to +\infty} r_k G_i^2(\bar{z}_k)\\
&\quad = \lim_{k \to +\infty} \frac{r_k G_i^2(\bar{z}_k) H_i(\bar{z}_k)}{H_i(\bar{z}_k)}\\
&\quad = \lim_{k \to +\infty} \frac{G_i(\bar{z}_k) y_i^k + r_k G_i^2(\bar{z}_k) H_i(\bar{z}_k)}{H_i(\bar{z}_k)}\\
&\quad = 0, \quad i \in I^c(\bar{z}_k) \cap I^1(\bar{z}).
\end{aligned}$$

It follows that

$$\lim_{k \to +\infty} r_k G_i^2(\bar{z}_k) (\nabla H_i(\bar{z}_k)^T d_k) = 0, \quad i \in I^c(\bar{z}_k) \cap I^1(\bar{z}). \tag{34}$$

Analogously, we have

$$\lim_{k \to +\infty} r_k H_i^2(\bar{z}_k) (\nabla G_i(\bar{z}_k)^T d_k) = 0, \quad i \in I^c(\bar{z}_k) \cap I^2(\bar{z}). \tag{35}$$

Taking the limit in (33) while noticing (22), (34) and (35), we see that (4) holds. The proof is complete.

Remark 3.1. Recently, Lin and Fukushima ([17]) showed that a weaker condition than the usually used second-order necessary condition of the corresponding subproblems involved in the respective methods (see, e.g. [10,13,14,24]) (together with a set of other conditions) can guarantee their scheme to a B-stationary point. This condition is described by the uniform lower boundedness of the Lagrangian functions of their relaxed problems on some tangent space. One can see from the proof of the first part (convergence to a strongly stationary point) of Theorem 3.2 that the assumption of second-order necessary condition of $(P(y^k, r_k))$ at $\bar{z}_k$ can also be replaced by a weaker condition similar to the one used in [17]: there exists $\alpha > 0$ such that for any d satisfying (28), (29) holds with its right-hand side replaced by $-\alpha d^T d$.

4. Numerical Experiments

First, we state the detailed procedures of the partial augmented Lagrangian method as follows.

ALGORITHM.

Step 1. Set $\beta > 1$, $0 < \gamma < 1$.

Step 2. Initialize the penalty parameter $r > 0$, the multiplier $y \in R^p$ and the starting point z_0.

Step 3. Solve the problem $(P(y, r))$ with the starting point z_0 by some method for ordinary constrained optimization problems and obtain a solution $\bar{z}$. If the value of the term

$$\max\Big\{|G_i(\bar{z})H_i(\bar{z})|, (-G_i(\bar{z}))^+, (-H_i(\bar{z}))^+, i = 1, \ldots, p, g_j^+(\bar{z}), \\ j \in J_1, |g_j(\bar{z})|, j \in J_2\Big\}$$

is small enough, where $u^+(z) = \max\{u(z), 0\}$, then stop and $\bar{z}$ is regarded as an approximate optimal solution. Otherwise, go to Step 4.

Step 4. If $\sum_{i=1}^p |G_i(\bar{z})H_i(\bar{z})| > \gamma \sum_{i=1}^p |G_i(z_0)H_i(z_0)|$, then set $r = \beta r$, $z_0 = \bar{z}$ and go to Step 3; otherwise, set $y_i = y_i + rG_i(\bar{z})H_i(\bar{z})$, $\quad i = 1, \ldots, p$, $z_0 = \bar{z}$ and go to Step 3.

We tested the methods proposed in [10,13,14,24] and this paper for MPCC on a number of problems. For the sake of convenience, we label these methods as FP, S, HR, HYZ and HYT, respectively (taking the first letter(s) of the author(s) of the corresponding method).

For all these methods, we use the starting point $z_0 = 0$ and the starting point for each of the subsequent subproblems is the solution of the last solved subproblem (warm start). All the methods use the stop rule that the constraint violation

$$\max\Big\{|G_i(\bar{z}^k)H_i(\bar{z}^k)|, (-G_i(\bar{z}^k))^+, (-H_i(\bar{z}^k))^+, i=1,\dots,p, g_j^+(\bar{z}^k), \\ j\in J_1, |g_j(\bar{z}^k)|, j\in J_2\Big\}$$

is less than $1.e-6$ or it is hard to reduce the constraint violation, where k is the index of the last problem solved. Each subproblem involved in these methods is solved by directly invoking the MATLAB (Version 6.1) subroutine (for constrained or unconstrained programs).

For the methods FP and S, we set the initial value of the parameter $\epsilon = 0.1$ and the update rule is $\epsilon_k = \epsilon_{k-1}/5$.

For the method HR, at each outer iteration, we solve the following subproblem:

$$\begin{aligned} \min \quad & f(z) + r\sum_{i=1}^{m} F_i(z)G_i(z) \\ \text{s.t.} \quad & G_i(z)\geqslant 0, \qquad H_i(z)\geqslant 0, \quad i=1,\dots,p, \\ & g_j(z)\leqslant 0, \quad j\in J_1, \\ & g_j(z)=0, \quad j\in J_2, \end{aligned}$$

where $r>0$ is the penalty parameter.

For both HR and HYZ, the initial value of the penalty parameter r is 1 and the update rule is $r_k = 10r_{k-1}$.

For the method HYT, we take $\beta = 10$, $\gamma = 0.25$ as recommended in [2] (for ordinary nonlinear programming). The initial multiplier y and initial penalty parameter r are set to 0 and 1, respectively.

The numerical tests consist of three tests for observation of computational performances of these methods.

TEST I.

We implemented the five methods on Example 6.1 in [25] and Problems 1–11 in [6], some of which are mathematical programs with nonlinear complementarity constraints. All these five methods worked well on these test problems. Some of the optimal values obtained by any one of these five methods are even better than those obtained in [6,20,21].

TEST II.

All the test problems in Test I are small sized. In order to see computational performances of the five methods on larger scale problems, we used some test problems from QPECgen developed by Jiang and Ralph (see [16]). This package can generate random quadratic programs with linear complementarity constraints, which are special cases of MPCC. Specifically, the following type of problems will be generated:

$$\begin{aligned} \min \quad & f(x,y) = 1/2(x,y)^T P(x,y) + c^T x + d^T y \\ \text{s.t.} \quad & F(x,y) = Nx + My + q \geqslant 0, \\ & y \geqslant 0, \\ & F^T(x,y)y = 0, \\ & g(x,y) = A(x,y) + a \leqslant 0, \end{aligned}$$

where $(x,y) \in R^{n_1+n_2}$, $N \in R^{n_2 \times n_1}$, $M \in R^{n_2 \times n_1}$, $q \in R^{n_2}$, $A \in R^{l \times (n_1+n_2)}, a \in R^l$.

x and y are called first level variable and second level variable, respectively. Some of the QPECgen parameters [except the problem dimensions (n_1, n_2, m) and the parameters second_{deg}, mix_{deg} and mono_M, which will be specified below] are set as follows: $\text{qpec}_{type} = 300, p = l$, $\text{cond}_P = 100$, $\text{scale}_P = \text{cond}_P$, $\text{convex}_f = 1$, $\text{symm}_M = 1$, $\text{cond}_M = 200$, $\text{scale}_M = 200$, $\text{first}_{deg} = 2$, $\text{tol}_{deg} = 1.e-6$, $\text{implicit} = 1$, $\text{rand}_{seed} = 0$, $\text{output} = 1$. Since we set $\text{convex}_f = 1$, the problems we tested are convex quadratic programs with linear complementarity constraints.

Dimensions of the problems we tested are: $(8,10,4)$, $(8,20,4)$, $(8,30,4)$, $(8,40,4)$, $(8,50,4)$, $(8,60,4)$, $(8,80,4)$, $(8,90,4)$, $(8,100,4)$, $(8,150,4)$, $(8,200,4)$, $(100,4,4)$, $(150,4,4)$, $(200,4,4)$, $(100,10,4)$, $(150,5,4)$.

Three groups of values for the parameters second_{deg}, mix_{deg} and mono_M were used for each of the above problem:

(a) $\text{second}_{deg} = 4$, $\text{mix}_{deg} = 2$, $\text{mono}_M = 1$;
(b) $\text{second}_{deg} = 4$, $\text{mix}_{deg} = 2$, $\text{mono}_M = 0$;
(c) $\text{second}_{deg} = 0$, $\text{mix}_{deg} = 0$, $\text{mono}_M = 1$.

We listed below the problems which the stated method “failed” to solve. [The method failed to generate a feasible solution or the optimal value obtained is much larger (usually more than 30%) than the one recommended by the QPECgen.] In the following, we will use notations (a)–(c), meaning parameter settings (a)–(c), respectively.

FP.

(a) $(100,10,4)$, $(150,4,4)$, $(150,5,4)$;
(b) $(8,40,4)$, $(8,50,4)$, $(8,60,4)$, $(8,100,4)$, $(8,150,4)$, $(8,200,4)$, $(150,4,4)$, $(200,4,4)$;
(c) $(100,10,4)$.

S.

(a) $(100,10,4)$, $(150,5,4)$;
(b) $(8,40,4)$, $(8,60,4)$, $(8,200,4)$, $(150,4,4)$, $(200,4,4)$, $(150,5,4)$;
(c) $(100,10,4)$.

HR.

(a) none;
(b) $(100, 10, 4)$;
(c) none.

HYZ.

(a) none;
(b) none;
(c) none.

[Only some optimal values obtained are larger but not much larger (usually no more than 5%) than the ones recommended by QPECgen.]

HYT.

(a) $(8, 50, 4)$, $(8, 80, 4)$, $(8, 90, 4)$, $(8, 100, 4)$, $(8, 150, 4)$, $(8, 200, 4)$;
(b) $(8, 80, 4)$, $(100, 10, 4)$;
(c) $(8, 40, 4)$, $(8, 80, 4)$, $(8, 90, 4)$, $(8, 150, 4)$, $(8, 200, 4)$, $(200, 4, 4)$.

It seems from Test II that

(i) HYT performs better FP and S for problems with parameter setting (b).
(ii) For parameter setting (a), if the dimension of the second level variable is slightly large (e.g. $\geqslant 50$), then HYT performs very poorly. However, if the first level variable is slightly large and the second level variable is small, it worked better. Meanwhile, FP and S may perform poorly when the dimension of the first level variable is large.
(iii) For parameter setting (c), FP and S usually perform better than HYT.
(iv) The methods HR and HYZ perform well on almost all the tested problems.

One more thing that is worth mentioning is that augmented Lagrangian method usually performs better than penalty methods in ordinary nonlinear programming (see, e.g. [2]), whereas the partial augmented Lagrangian method HYT seems to perform more poorly than the partial penalty method HR when applied to MPCC. Up to now, we do not know the reason.

TEST III.

To further see the performances of the methods HR and HYZ, both of which performed well in Test II, we tested the two methods on larger QPEC problems. The notations and parameters were set as in Test II unless stated otherwise.

The dimensions of the tested problems are $(58, 50, 4)$, $(108, 100, 4)$, $(158, 150, 4)$, $(208, 200, 4)$ with parameter settings (a)–(c) as in Test II.

The HR failed in problems $(58, 50, 4)$, $(158, 150, 4)$, $(208, 200, 4)$ with any one of the three parameter settings (a)–(c).

For each of these tested problems, HYZ succeeded in generating a solution whose optimal value is close to (no more than 5%) the recommended one [except for the problem $(158, 150, 4)$ with parameter settings (a) and (c), the optimal values obtained are about 15% larger than the recommended ones]. In addition, the CPU time consumed by HYZ is much shorter than that of HR for every suceeded problem.

Finally, we note that we believe the numerical performances observed in the above experiments can not completely reflect the advantages and disadvantages of each tested method since our tests are based on the direct invoking of MATLAB subroutines to solve each subproblem involved in the respective methods in spite of the fact that specific effective and efficient methods might be designed to solve the respective subproblems.

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